Behavior of realized volatility and correlation in exchange markets

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ABSTRACT

We study time-varying realized volatility and related correlation measures as proxies for the true volatility and correlation. We investigate measures of Two-Scale realized Absolute Volatility (*TSAV*) and correlation (*TSACORxy*) which are helpful to cope effectively with the problem of market microstructure effects at very high frequency financial time series. The measures are constructed based on subsampling and averaging method so that they possess rather less bias even in presence of market microstructure noise. Absolute transformation of return values has been proved in literature to be more robust than squared transformation when considering large values. With respect to some stylized facts of markets, realized squared correlation does not display dynamic behavior. Motivated by robustness of realized absolute volatility, we study an alternative measure of correlation, built on absolute-transformed volatility. This measure of correlation exhibits experimentally some dynamics and hence some predictability capability on minute-by-minute frequency exchange market data. We show that the distribution of realized correlation series computed based on *TSACORxy* tends to comply a rightward asymmetric shape implying that upside co-movements are greater than downside ones. Moreover we study the association between realized volatility and correlation. According to the two-scale measure, our findings empirically suggest that when returns in Euro/USD exchange rate are highly volatile, the relation between Euro/USD and Euro/GBP exchange markets is strong, and when Euro/USD calms down, the relationship relaxes.

Key words: *Realized Volatility and Correlation, Long Memory, Scaling Law, Self-Similarity Dimension, Market Microstructure Effects.*
JEL Classifications: C14, C51, C58, F31, G15

1. INTRODUCTION

Measuring and forecasting financial volatility is of crucial importance to asset and derivative pricing, asset allocation and risk management. Hence, financial economists have been intrigued by the very high precision with which volatility can be estimated under the diffusion assumption routinely invoked in theoretical work. Although most textbook models assume volatilities and correlations to be constant, it is widely recognized among both finance academics and practitioners that they vary importantly over time, with persistent dynamics. Furthermore, their fluctuations display substantial volatility persistence (Andersen et al., 1999a). The basic insight follows from the observation that precise estimation of diffusion

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The authors gratefully acknowledge the helpful and substantive comments of Editors and Referees on concepts related to this paper.

volatility does not require a long calendar span of data; rather, volatility can be estimated arbitrarily well from an arbitrarily short span of data, provided that returns are sampled with sufficient frequency. This contrasts sharply with precise estimation of the drift, which generally requires a long calendar span of data, regardless of the frequency with which returns are sampled. Consequently, the volatility literature has steadily progressed toward the use of higher frequency data (Andersen et al., 1999a). Most of what we have learned from this burgeoning literature is based on the estimation of parametric ARCH or stochastic volatility models (SV) for the underlying returns. However, the validity of such volatility measures generally depends upon specific distributional assumptions (Andersen et al., 2001a).

The use of higher frequency data, now increasingly available, has also been concurred the emerging theories emphasizing the advantages of the so-called realized volatility and realized power variation as well as correlation estimators.

It has been recognized that volatility is inherently unobserved, and evolves stochastically through the time. Volatility models are cast either in discrete time or continuous time. It is clear, however, that the trading and pricing of securities in many of today's liquid financial asset markets is evolving in a near continuous fashion throughout the trading day. As such, it is natural to think of the price and return series of financial assets as arising through discrete observations from an underlying continuous time process (Andersen et al., 2006). Any log-price process subject to a no-arbitrage condition and weak auxiliary assumptions will constitute a semi-martingale that may be decomposed into a locally predictable mean component and a martingale with finite second moments. Andersen et al. (2006) argue that the return variance is approximately equal to the expected squared return innovation. This suggests that we may be able to measure the return volatility directly from the squared return observations. However, this feature is not of much direct use as the high frequency returns have a large idiosyncratic component that induces a sizeable measurement error into the actual squared return relative to the underlying variance. In reality, there is a definite lower bound on the return horizon that can be used productively for computation of the realized volatility, both because we only observe discretely sampled returns and, more important, market microstructure frictions on intradaily level such as discreteness of the price grid, asymmetries in information, transaction costs, bid-ask spreads, lunch-time effects, and U-shape volatility of trading volume over the day induce gross violations of the semi-martingale property at the very highest return frequencies. This implies that we typically will be sampling returns at a high frequency that leaves a non-negligible error term in the estimate of integrated volatility.

By construction, the realized squared volatility is an observed proxy for the underlying quadratic variation and the associated measurement errors are uncorrelated. This suggests a straightforward approach where the temporal features of the series are modelled through standard time series techniques, letting the data guide the choice of the appropriate distributional assumptions and the dynamic representation. This is akin to the standard procedure for modelling macroeconomic data where the underlying quantities are measured (most likely with a substantial degree of error) and then treated as directly observed variables (Andersen et al., 2006).

We proceed under the convenient assumption that we are dealing with correctly specified models and the associated full information sets, so that the conditional first and second moments are directly observable and well specified. It is useful to think of the returns as arising from an underlying continuous-time process. In particular, suppose that this underlying model involves a continuous sample path for the (logarithmic) price process. Under general assumptions, the price process may then be written in standard stochastic differential equation form as

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) \tag{1.1}$$

where time $t \geq 0$, $p(t)$ is a price at (t) and indeed a semimartingale or a Brownian semimartingale, $\mu(t)$ denotes the drift, $\sigma(t)$ refers to the or spot volatility, and $W(t)$ denotes a standard Brownian motion. The σ is called the spot volatility process and μ the mean or risk premium process. Intuitively, over infinitesimal small time intervals, i,

$$r(t,i) \equiv p(t) - p(t-i) \approx (t-i)\cdot i + \sigma(t-i) iW(t)$$

where $iW(t) \equiv W(t) - W(t\text{–}i) \sim N(0,i)$. Of course, for $i = 1$, and constant drift, $\mu(\tau) \equiv \mu_{t\,|\,t-1}$, and volatility, $\sigma(\tau) \equiv \sigma_{t\,|\,t-1}$, for $t - 1 < \tau \leq t$, this reduces to the discrete time return decomposition

$$r_t = \mu_{t\,|\,t-1} + \epsilon_t = \mu_{t\,|\,t-1} + \sigma_{t\,|\,t-1} z_t \tag{1.2}$$

where z_t denotes an i.i.d. with mean zero, variance one, serially uncorrelated disturbance (white noise) process, and r_t, the discretely sampled return process, which is readily decomposed into an expected conditional mean return and an innovation, where the latter may be expressed as a standardized white noise process scaled by the time-varying conditional volatility. The drift, μ_t, and instantaneous volatility, $\sigma(t)$, for the continuous time model in (1.1) need not be constant over the $[t\text{–}1, t]$ time interval, resulting in the general expression for the one-period return,

$$r(t) = p(t) - p(t-1) = \int_{t-1}^{t} \mu(s)ds + \int_{t-1}^{t} \sigma(s)dW(s) \tag{1.3}$$

Similarity between this representation and the previous one-period return for the discrete-time model in (1.2) is clear. The conditional mean and variance processes in the discrete formulation are replaced by the corresponding integrated realizations of the (potentially stochastically time-varying) mean and variance process over the following period, with the return innovation driven by the continuously evolving standard Brownian motion. Intuitively, the volatility for the continuous-time process in (1.1) over $[t\text{–}1, t]$ is intimately related to the evolution of the diffusive coefficient, $\sigma(t)$, which is also known as the spot volatility. In fact, given the i.i.d. nature of the return innovation governed by the Brownian motion process, the return variation should be related to the cumulative (integrated) spot variance. It is, indeed, possible to formalize this intuition: the conditional return variation is linked closely and – under certain conditions in an ex-post sense – equal to the so-called Integrated Power Volatility of order r (*IPV*),

$$IPV(t) = \int_{t-1}^{t} \sigma^{r}(s)ds \tag{1.4}$$

as the sampling frequency increases. In other words, the estimation error of the realized power volatility diminishes. Here, r denotes a positive value.

In order to get a discrete approximation to the $IPV(t)$, Barndorff-Nielsen and Shephard (2003) propose the realized power variation of order r, $\langle \widehat{X,X} \rangle_{RP}$, as a proxy for the true integrated power volatility as

$$\langle \widehat{X,X} \rangle_{RP} = \sum_{t_i}^{T} \left| Y_{t_{i+1}} - Y_{t_i} \right|^{r} \tag{1.5}$$

where i=1,...,n is ith intraday observation with an integer n and r is a positive value. Here, Y_{t_i} is a price observed on day t at time i which follows the price process (1.1), and $Y_{t_{i+1}} - Y_{t_i}$ gives a return of high frequency prices which follows (1.2). Definitely where r=2 in (1.5), the realized power variation would approximate the so-called Realized Volatility as introduced by Andersen and Bollerslev (1998) and Andersen et al. (2001b). In this case, the result of realized power variation considerably strengthens the quadratic variation result that realized quadratic variation converges in probability to Integrated Volatility, $IV(t)=\int_{t-1}^{t}\sigma^2(s)ds$ (Barndorff-Nielsen and Shephard, 2003). Realized power variation theory covers also Realized Absolute (RA) variation in which case r=1 (Barndorff-Nielsen and Shephard, 2003).

According to (1.4) and Theorem 1 in Barndorff-Nielsen and Shephard (2003), the realized power variation of order r, $\langle \widehat{X,X} \rangle_{RP}$, computed from the highest frequency data (as $n \rightarrow \infty$) should provide the best possible estimate for the integrated power volatility. However, this is not the general point of view adopted in the empirical finance literature. In practice, sampling at the very high frequency (for example higher than 5 minute frequency) leads to a well-known bias problem due to market microstructure noise (Zhou, 1996 and Andreou and Ghysels, 2002). It is generally accepted that the return process should not be sampled too often (Zhang et al., 2005); since the market microstructure effects intervene to cause noise and hence a bias of estimation due to for example the bid-ask bounce, when applying very high frequency data in real situations.

To cope with the problem of market microstructure effects when approximating realized power variation, a successful alternative approach[1] called Two-Scale Realized Volatility (TSRV), based on a subsampling and averaging procedure has been proposed by Zhang et al. (2005). Their device, constructed based on a squared transformation of returns, is model-free too and takes advantage of the rich sources of tick-by-tick data, and to a great extent corrects for the adverse effects of microstructure noise on volatility estimation. However, on one side, according to the literature, for example Ding et al. (1993), Forsberg and Ghysels (2005), Andersen et al. (2006) and Ghysels et al. (2006), a squared transformation of returns in a TSRV model in turn reinforces jumps to appear in volatility series as large values. Thus, this model seems theoretically not to be robust against jumps, meanwhile construction of volatility based on realized power variation with absolute transformation is somewhat robust to rare jumps (Barndorff-Nielsen and Shephard, 2004a), in particular in case of r=1 (or Realized Absolute variation). On the other side, their approach can be seen as a specific case of what we are trying to explain; since realized volatility is seen as a specific case of realized power variation as stated above. Therefore, in this regard, we generalize the TSRV approach on the broader realized power variation. In summary, realized power variation suffers from microstructure noise in particular in the form of higher bias, and TSRV suffers from jumps in the form of higher variance at higher frequencies.

To solve the problem of the market microstructure effects, inspired by the TSRV modeling of realized volatility and the robustness of absolute transformation of power variation, the Two-Scale realized Power Volatility (TSPV) measure is assumed to be consistent for integrated power volatility (IPV), (1.4), at very high frequency. The TSPV estimator of volatility should be robust against jumps, since it is based on absolute transformation inspired by realized power variation, and should be unbiased against microstructure noise inspired by two-scale procedure, since it is built on a bias-corrector method.

[1] There are many approaches to correct the microstructure noise, including for example a kernel-based correction introduced by Zhou (1996), an optimal sampling introduced by Bandi and Russell (2006), a moving average filter introduced by Maheu and McCurdy (2002), an autoregressive filter introduced by Bollen and Inder (2002), and of course a subsampling and averaging approach introduced by Zhang et al. (2005). However, it has been experimentally shown by Ghysels and Sinko (2006) that the subsampling and averaging class of estimators predicts volatility the best among microstructure noise correctors.

A realized correlation estimator is also drawn based on the TSPV estimator which seems to be more sound. An observable correlation model which does not fail to describe stylized facts as much as possible, observed in financial time series, is here desired.

The paper is organized as follows: Starting with realized squared volatility, in section 2, we construct realized volatility and correlation measures. In section 3, applying minute-by-minute frequency exchange rate data, the measures are evaluated by simulation. In section 4, some distributional and dynamic properties of measures are experimentally studied. It is shown that the volatility series are far from a normal distribution. However, in a relative sense, absolute based volatility measures are closer to normal distribution, because they react less sensitively to jumps. Two self-similar dimensions which statistically indicate regularity and dynamic properties of measures are investigated. The distributional and dynamic behaviors of correlation measures are also compared. In section 5, relationship between realized volatilities and realized correlations is studied. In section 6, the results are summarized and discussed.

2. REALIZED VOLATILITY AND CORRELATION MEASUREMENTS

Merton (1980) showed that the integrated volatility of a Brownian motion (1.3) and hence (1.4) over a fixed interval can be approximated to an arbitrary precision using the sum of intraday squared returns, provided the data are available at a sufficiently high sampling frequency. More recently Andersen and Bollerslev (1998) and Andersen et al. (2001b), applying the quadratic variation theory, generalized this result to the class of special (finite mean) semimartingales. This class encompasses processes used in standard arbitrage-free asset pricing applications, such as, Ito diffusions, jump processes, and mixed jump diffusions. In fact, under such conditions, the sum of intraday squared returns converges to the integrated volatility of the prices, as the maximal length of returns goes to zero, allowing us, in principle, to construct an error free estimate of the actual volatility over a fixed-length time interval (Engle and Bollerslev, 1986). The standard definition for an equally spaced returns series of the Realized Squared (RS) volatility over a time interval is

$$\langle \widehat{X,X} \rangle_{RS} = \sum_{t_i}^{T} \left(Y_{t_{i+1}} - Y_{t_i} \right)^2 \tag{2.6}$$

where $\langle \widehat{X,X} \rangle_{RS}$ is the estimated realized squared volatility, and Y_{t_i} with $0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$, is an observed log transformed high frequency price of a financial asset.

Ding et al. (1993) found that not only there is substantially more correlation between absolute returns than returns themselves, but the power transformation of the absolute return, $|Y_{t_{i+1}} - Y_{t_i}|^r$, also has quite high autocorrelation for long lags. It is possible to characterize $|Y_{t_{i+1}} - Y_{t_i}|^r$ to be long memory and this property is strongest when r is around 1. This result appears to argue against ARCH type specifications based upon squared returns. Granger and Sin (2000) treated observed absolute return as a measure of risk against unobserved (conditional) conventional variance and explored its forecastability. They applied models using two measures to three stock indices, and reported that the model applied to absolute measure largely outperforms the alternative model applied to variance both in-sample goodness of fit and post-sample forecastability. The distribution theory for quadratic variation under the continuous sample path assumption has been extended to cover cumulative absolute returns raised to an arbitrary power. The leading case involves cumulating absolute returns of high-frequency. These quantities display improved robustness properties relative to realized squared volatility as the impact of jumps are mitigated. Limit theorems were also derived for measures, called realized power variation, over a fixed interval of time, as the number of high frequency increments goes to infinity by Barndorff-Nielsen and Shephard (2003). Indeed, they presented a theory, in particular, for the use of sums of absolute returns for example, the analysis of volatility models using high frequency information and turbulence and image analysis. Based on a simulation with different number of daily observations, they found that the realized power variation version of the statistic has much better finite sample behavior, while the realized quadratic variation behaves quite poorly. Measures built on absolute values are less sensitive to possible large movements in high frequency data. There is evidence that if returns do not possess fourth moments then using absolute values rather than squares would be more reliable (Barndorff-Nielsen and Shephard, 2003).

As mentioned above, Barndorff-Nielsen and Shephard (2003) introduced the estimator based on power returns which they call Realized Power (RP) measure, $\widehat{\langle X,X\rangle}_{RP}$, in the form of (1.5) with a positive r and the same previous notation. Again Barndorff-Nielsen and Shephard (2004a) and Barndorff-Nielsen et al. (2004b) extended the estimator of Realized Power (RP) measure to the wider versions, called realized bipower, multipower, normalized and generalized multipower variations.

In order for dealing with microstructure noise resulting, for example, to the bias problem of sampling at a very high frequency and for increasing accuracy of measure, Zhang et al. (2005) have introduced the Two-Scale Realized Volatility estimator (TSRV), which combines the realized squared volatility estimators from two time scales. The volatility estimator $\widehat{\langle X,X\rangle}_{TSRV}$ combines the sum of squared estimators from two different time scales; $\widehat{\langle X,X\rangle}_{avg}$ from the returns on a slow time scale, whereas $\widehat{\langle X,X\rangle}_{RS}$ is computed from the returns on a fast time scale using the latter as a means for bias-corrector of the subsampling and averaging based measure. The $\widehat{\langle X,X\rangle}_{avg}$ estimator is constructed based on subsampling and averaging procedure.

Motivated by superiority of realized power volatility measure (RP) in relative less variation, on one hand, and benefits of subsampling and averaging frequencies procedure in the Two-Scale squared Realized Volatility (TSRV) for dealing with microstructure noise, on the other hand, we extended Safari and Seese (2008) a realized power volatility measure to the Two-Scale realized Power Volatility (TSPV) estimator for Integrated Power Volatility (1.4). The bias of the estimator TSPV can be lessened by the averaging on samples. The TSPV has less variation relative to TSRV, in particular where r=1, since it is less sensitive to the large points in a given time series than squared values.

In order to prescribe the TSPV estimator, at first the subsampling method has to be shortly illustrated. The method looks like the Jackknife method. The goal of reducing bias of estimation for a statistic in two methods seems the same. Efron and Gong (1983) conclude that like the bootstrap, the Jackknife can be applied to any statistic that is a function of n independent and identically distributed variables. It performs less well than the Bootstrap but requires less computation. The Jackknife resamples the statistic at the n points. Efron and Gong (1983) state that the Jackknife is almost a Bootstrap itself. Goncalves and Meddahi (2005) propose bootstrap methods for statistics evaluated on high frequency data such as realized volatility.

The subsampling method includes two time scales, one fast and one slow. Let $g^{(k)}$ be a disjoint subset of the full set of observation times with union g and n be the number of sampling intervals over $[0,T]$. Here an averaging estimator is defined based on selecting a number of subgrids of the original grid of observation times, $g = \{t_0 ,..., t_n\}$, and then on averaging the estimators derived from the subgrids. We suppose that the full grid g, $g=\{t_0 ,..., t_n\}$, is partitioned into K non-overlapping subgrids $g^{(k)}$, $k=1,...,K$. It is easy to define an average estimator according to the subgrids. The average estimator, $\langle \widehat{X,X} \rangle_{avg}$, is defined on a slow scale estimator as

$$\langle \widehat{X,X} \rangle_{avg} = \frac{1}{K} \sum_{k=1}^{K} \sum_{t_{i+1},t_i \in g^{(k)}} \left| Y_{t_{i+1}} - Y_{t_i} \right|^r \tag{2.7}$$

and, in a special case when the sampling points are regularly allocated, as

$$\langle \widehat{X,X} \rangle_{avg} = \frac{1}{K} \sum_{t_{i+1},t_i \in g^{(k)}} \left| Y_{t_{i+1}} - Y_{t_i} \right|^r \tag{2.8}$$

where $g^{(k)}$ is a subset of the full set of observations.

We then estimate the Two-Scale realized Power Volatility (TSPV), $\langle \widehat{X,X} \rangle_{TSPV}$, by

$$\langle \widehat{X,X} \rangle_{TSPV} = \langle \widehat{X,X} \rangle_{avg} - \frac{\bar{n}}{n} \langle \widehat{X,X} \rangle_{RP} \tag{2.9}$$

and when a small-sample adjustment $(1-{}^{\bar{n}}/_{n})^{-1}$ is needed, by

$$\langle \widehat{X,X} \rangle_{TSPV} = \left(1 - \frac{\bar{n}}{n}\right)^{-1} \left(\langle \widehat{X,X} \rangle_{avg} - \frac{\bar{n}}{n} \langle \widehat{X,X} \rangle_{RP} \right) \tag{2.10}$$

where $\langle \widehat{X,X} \rangle_{RP}$ is simply computed in (1.5) on a fast scale. A profound theoretical justification for application of the subsampling method in the area of realized volatility can be found in Zhang et al. (2005).

The benefits of a high frequency realized volatility approach for measuring, modelling and forecasting univariate volatilities may motivate one to construct similarly realized covariance and correlation. By the theory of realized variation, Andersen et al. (2001a) and Andersen et al. (2001b) also derived realized standard deviation, $RS_{std} = (\langle \widehat{X,X} \rangle_{RS})^{½}$; logarithmic standard deviation, RS_{lstd} = ½.log$\langle \widehat{X,X} \rangle_{RS}$; covariance, $RCOV_{xy}=\sum_{t_i}^{T} (Y_{t_{i+1}} - Y_{t_i})_x \cdot (Y_{t_{i+1}} - Y_{t_i})_y$; and realized squared-based correlation, $RSCOR_{xy}$, as follow

$$RSCOR_{xy} = RCOR_{xy} / (RS_{std,x} \cdot RS_{std,y}) \tag{2.11}$$

where x and y are two assets or high frequency time series.

If the idea of an extension of high the frequency realized volatility approach to the measures of covariance and correlation is already convincing, then the extension of absolute-based realized volatility to absolute-based realized correlation would apparently seem a promising of this idea. Also the subsampling and averaging procedure, in order to enhance precision and to reduce microstructure noise problems and hence the bias problem, may help to realize the purpose of constructing time-varying realized covariance and correlation which are more robust to jumps and may be more predictable. Squared transformation instead of absolute one for constructing a measure of correlation might lead to overestimation in correlation. Thus, based on absolute transformation, in Safari and Seese (2008) we derived absolute-based realized standard deviation, $RA_{std}=(\langle \widehat{X,X} \rangle_{RA})^{½}$ (note $\langle \widehat{X,X} \rangle_{RA}=\langle \widehat{X,X} \rangle_{RP}$ where r = 1), logarithmic standard deviation, RA_{lstd}=½.log$\langle \widehat{X,X} \rangle_{RA}$, covariance, $RCOV_{xy}=\sum_{t_i}^{T} (Y_{t_{i+1}} - Y_{t_i})_x \cdot (Y_{t_{i+1}} - Y_{t_i})_y$, and realized absolute correlation, $RACOR_{xy}$, as follows

$$RACOR_{xy} = RCOR_{xy} / (RA_{std,x} \cdot RA_{std,y}) \tag{2.12}$$

where x and y are two assets or high frequency time series. We note that covariance remaines the same. Also these measures could simply be extended to measures based on subsampling and averaging procedure, so that we will have $TSAV_{std}=(\langle\widehat{X,X}\rangle_{TSAV})^{½}$ (note that r=1), $TSAV_{lstd}$=½.log$\langle\widehat{X,X}\rangle_{TSAV}$, and $TSCOV_{xy}$, as follows

$$TSCOV_{xy} = RCOV_{xy,avg} - \frac{\bar{n}}{n}\ RCOV_{xy,all} \tag{2.13}$$

where $RCOV_{xy,all}$ is the same as $RCOV_{xy}$, built on the all scale (the full grid). Here $\langle\widehat{X,X}\rangle_{TSAV}$ is the same as $\langle\widehat{X,X}\rangle_{TSPV}$ where r=1. In case of need for the small-sample adjustor, the term can be multiplied by the right hand of (2.13). Here $RCOV_{xy,avg}$ can be computed by

$$RCOV_{xy,avg} = \frac{1}{K}\sum_{k=1}^{K}\sum_{t_{i+1},t_i\in g^{(k)}}(Y_{t_{i+1}}-Y_{t_i})_x(Y_{t_{i+1}}-Y_{t_i})_y \tag{2.14}$$

The Two-Scale Absolute Correlation, $TSACOR_{xy}$ (based on r=1), is computed as follows

$$TSACOR_{xy} = TSCOV_{xy} / (TSAV_{std,x} \cdot TSAV_{std,y}) \tag{2.15}$$

where $TSACOR_{xy}$ denotes the time-varying and instantaneous conditional correlation between the returns of two time series x and y.

In the next section, we evaluate the asymptotic convergence and unbiasedness of the estimators by simulations. The RP and TSRV estimators are compared with TSPV just in a special case of r=1 for RP and TSPV and of r=2 for TSRV. Therefore, RA and TSRV are considered as benchmarks for TSAV.

3. SIMULATION EXPERIMENTS

The GARCH(1,1) model has appeared as a base for modeling volatility in financial time series, as it tends to provide a simple estimation to the main statistical features of the return series across a wide range of assets. For the simulation part of the present work, we advocate Andersen and Bollerslev (1998) and Andersen et al. (1999b) and establish the diffusion foundation for analysis. Following Nelson (1990) and Drost and Werker (1996), the continuous-time diffusion limit of the GARCH(1,1) model is given by

$$dp_t = \sigma_t\, dW_{1,t} \tag{3.16}$$

$$d\sigma_t^2 = \theta(\omega - \sigma_t^2)dt + (2\lambda\theta)^{1/2}\sigma_t^2 dW_{2,t} \tag{3.17}$$

where $W_{1,t}$ and $W_{2,t}$ denote independent standard Brownian motions. According to Drost and Werker (1996) the discretely sampled returns from the continuous-time process defined by Eqs. (3.16) and (3.17), satisfy the weak GARCH(1,1) model

$$\sigma^2_{(m),t} = \varphi_m + \alpha_m r^2_{(m),t-1/m} + \beta_m \sigma^2_{(m),t-1/m} \tag{3.18}$$

with m observations per day t, where $\sigma^2_{(m),t} \equiv P_{(m),t-1/m}\ (r^2_{(m),t})$ denotes the best linear predictor of $r^2_{(m),t}$. Note that here in this paper $r^2_{(m),t}$ has different alternatives defined previously in (1.5), (2.6) and (2.10). The relationship between the discrete-time parameters φ_m, α_m, and β_m with the continuous-time parameters ω, θ and λ may be obtained in closed form, as outlined by Drost and Werker (1996). Hence, in this weaker interpretation a GARCH(1,1) specification for any discrete frequency is compatible with the diffusion in Eqs. (3.16) and (3.17), and in this sense the setting provides a coherent framework for analysis of the model forecasts at different sampling intervals. Now, following Baillie and Bollerslev (1992) the h-period linear projection from the weak GARCH(1,1) model with returns that span $1/m$ day(s) is conveniently expressed as

$$P_{(m),t}(r^2_{(1/h),t+h}) = P_{(m),t}\left(\left[\sum_{j=1,\dots,mh} r_{(m),t+j/m}\right]^2\right) = \sum_{j=1,\dots,mh} P_{(m),t}(r^2_{(1/h),t+j/m})$$

$$= \sum_{j=1,\ldots,mh} [\sigma^2_{(m)} + (\alpha_m + \beta_m)^j (\sigma^2_{(m),t} - \sigma^2_{(m)})]$$

$$= mh\sigma^2_{(m)} + (\alpha_m - \beta_m)[1 - \alpha_m - \beta_m^{mh}] \times [1 - \alpha_m - \beta_m]^{-1} (\sigma^2_{(m),t} - \sigma^2_{(m)}) \quad (3.19)$$

where $\sigma^2_{(m)} \equiv \varphi_m(1-\alpha_m-\beta_m)^{-1}$.

As for market microstructure noise, ϵ, advocated by Barndorff-Nielsen et al. (2004b), Bandi and Russell (2005), Zhang et al. (2005) and Hansen and Lunde (2006) and recalling our assumptions about price and return processes, we assume that it follows a Gaussian process and is small. We assume a pure noise (i.e., noise is i.i.d. and independent with the efficient price). Specifically, we set $(E\epsilon^2)^{½}$=0.005, i.e., the standard deviation of the noise is 0.05% of the value of the variable of interest.

Like Andersen and Bollerslev (1998) and Andersen et al. (1999b) our theoretical assessment of performance of the discrete-time GARCH(1,1) approximation in Eq. (3.19) for predicting the subsequent realized volatility models defined by the stochastic volatility diffusion in Eqs. (3.16) and (3.17) rely on numerical means. More specifically, sample-path realizations of the underlying stochastic volatility diffusion are obtained via simulation using an Euler scheme.

Based on our daily real world data sample of Euro/USD exchange rate from June 1, 2006 to August 23, 2007, we estimate the parameters of continuous-time GARCH(1,1) models (3.16) and (3.17) equal to θ=0.0241 (Std Error=0.0128, T stat.=9.607), ω=3.3e-007 (Std Error=3.1e-007, T stat.=7.003), and λ=0.8325 (Std Error=0.0430, T stat.=27.014) with R^2=0.692 by MLE parameter estimation. The GARCH parameters are fixed at the values obtained from maximum likelihood estimation based on real daily observations of the Euro/USD exchange rate for simulation. Random variables are generated by MATLAB. For generating data, we assume 252 working days a year as usual and generate data at different frequencies according to Table 3.1. The simulations are based on 5 years of data samples and 8,000 sample paths (realizations). For all three alternative estimators, we assume equally distance sampling interval.

The values are transformed into logarithm form. After simulations the residuals are standardized in further estimations. The results of Monte Carlo simulation in terms of RMSE and bias in Table 3.1 show how the estimators converge to the integrated variation across frequencies when the sampling interval is going to diminish. Comparing the rows reveals asymptotic convergence in small sample distribution. Moreover, the following table shows a different behavior of estimators.

As expected by the theories of realized volatility and realized power variation, the variance of all estimators diminishes as the frequency increases. Therefore, all measures converge in terms of RMSE. This implies that the estimators are consistent for the targets, i.e., Integrated Volatility (in our special case of order 2 for TSRV, i.e., r=2) and Integrated Power Volatility (in our special case of order 1 for RA and TSAV, i.e., r=1). Hence, the estimators converge asymptotically as the frequency increases. This convergence is consistent with Zhang et al. (2005) and Barndorff-Nielsen and Shephard (2003). A comparison between estimators gives some information. There are obvious differences between the estimated RMSE errors of different estimators, since the estimators are converging in different rates. In fact, absolute based estimators converge faster in terms of RMSE. Differences in convergence rates are akin to the fact that the absolute based estimators are inherently somewhat immune against jumps in a relative sense. Consistent with Zhang et al. (2005), the subsampling and averaging method leads to a difference between RA and TSAV in terms of variation.

Frequency at every	TSRV		RA		TSAV	
	RMSE	Bias	RMSE	Bias	RMSE	Bias
60 min.	0.3826	0.0083	0.3602	0.0213	0.3519	0.0079
30 min.	0.3815	0.0072	0.3486	0.0214	0.3483	0.0057
15 min.	0.3503	0.0060	0.2772	0.0215	0.2731	0.0053
5 min.	0.3103	0.0031	0.1882	0.0230	0.1863	0.0025
1 min.	0.2907	0.0025	0.1036	0.0236	0.1034	0.0019
30 sec.	0.2599	0.0019	0.0765	0.0239	0.0704	0.0015
15 sec.	0.1323	0.0015	0.0432	0.0243	0.0430	0.0009
10 sec.	0.0815	0.0012	0.0112	0.0245	0.0109	0.0006
5 sec.	0.0117	0.0009	0.0039	0.0248	0.0038	0.0004
1 sec.	0.0095	0.0006	0.0003	0.0249	0.0002	0.0002

Table 3.1 Results of simulation (values*1000).

Consistent with the literature, the table simply shows that realized power volatility of order 1 (RA) is not an unbiased estimator of realized power variation as the frequency increases. Even the bias of the estimator is increasing across the frequencies caused by market microstructure frictions. From the table based on simulation, we find that the bias grows almost less than linearly in the number of intraday observations, when we consider RA estimator. This finding suggests that market microstructure noise is almost a linear direct function of observations or frequencies. This condition, however, is somewhat different around 5 minute frequency. Nonetheless, the bias of both subsampling and averaging based estimators converges to zero as the frequency increases. This is consistent with the theory mentioned above. Both TSRV and TSAV estimators are an unbiased estimator for realized volatility and realized power volatility. Meanwhile the bias of estimators can be compared. Considering both bias and variation of the estimators, TSAV estimates its true integrated power volatility (IPV) consistent and unbiased relative to others.

In the following section, distributional and dynamic properties of measures will experimentally be compared. Since there exists no two-scale realized squared correlation, we compare the results of measures with realized squared based correlation.

4. EMPIRICAL BEHAVIORS OF MEASURES

4.1. Data and Facilities

The empirical evidence suggests that daily realized volatility serves as a simple, yet effective, aggregator of the volatility information inherent in the intraday data (Andersen et al., 2006). For this section, our empirical analysis is based on returns of Euro/USD and Euro/GBP exchange rates at every 1 minute frequency. Our sample time series cover a period from June 1, 2006 to August 23, 2007. Both exchange rates are considered as a market with a high degree of liquidity and very active. We define return of an exchange rate by $Y_{t_{i+1}} - Y_{t_i} = \log(Y_{t_{i+1}}) - \log(Y_{t_i})$, which is the return from holding the currencies at time t_i to time t_{i+1}, where Y_{t_i} is the observed exchange rate value.

All computations and estimations in this work have been facilitated by the use of the software *R*, a system for statistical computation and graphics, and of the libraries therein including e1071, fBasics, tseries, and KernSmooth[2].

Statistic	Euro/USD	Euro/GBP
Minimum	-5.58e-03	-6.07e-03
Maximum	5.61e-03	5.70e-03
Mean	1.31e-07	-4.78e-08
Median	0.00e+00	0.00e+00
Sum	5.72e-02	-2.09e-02
Variance	3.88e-08	1.92e-07
Skewness	1.15e-01	-2.21e-02
Kurtosis	1.31e+01	3.97e-01
Jarque-Bera test	2.2e-16	2.2e-16

Table 4.2 Basic statistics and tests for return time series in exchange market.

Some more important descriptive statistics of our time series are contained in Table 4.2. Positive mean of return in Euro/USD explains an average positive return trend. In particular, an excess kurtosis with positive skewness in Euro/USD, and low kurtosis with negative skewness in Euro/GBP obviously show our time series depart from normality. Leptokurtosis in returns of Euro/USD is a sign of heavy tail in its distribution. This implies that there is a higher probability for extreme events than in data that is normally distributed. Negative coefficient of skewness for Euro/GBP (-0.022) series describes that our probability density function is negatively skewed. Therefore the distribution is asymmetric to the left side. However, the skewness coefficient for Euro/USD (0.115) indicates an asymmetry to the right side. Jarque-Bera test[3] for normality simply reveals that the time series with p-value equal to 2.2e-16 do not form a normal distribution.

4.2. Distributional Properties of Volatilities and Correlations

Considering the fact that volatility is now effectively observable and measurable, based on squared or absolute values and subsampling procedure, we can characterize their distributional properties with relying on conventional statistical procedures. Then, comparison of empirical distributions of different measures can be simply implemented.

Time series of realized volatility measures calculated based on (2.6), (1.5), and (2.10) with r=1 are depicted in Figure 4.1. Actually the figure unveils that volatility, constructed by all realized measures, is time-varying. This is in contrast to the conventional approach which views the volatility as constant.

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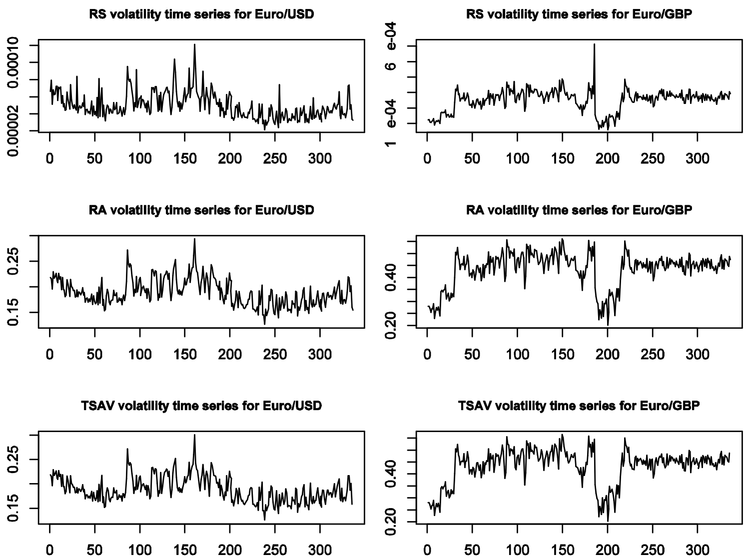
[2] More information about included packages, documents and downloading source codes can be found on: http://www.r-project.org.

[3] Note that the Jarque-Bera test of normality is likely the most widely used procedure for testing normality of economic time series returns. The algorithm provides a joint test of the null hypothesis of normality in that the sample skewness equals zero and the sample kurtosis equals three.

Figure 4.1 Time series of realized volatility measures constructed based on squared, absolute, and two-scale absolute transformations. They show daily volatility series for Euro/USD and Euro/GBP. Evidently volatility is viewed time-varying.



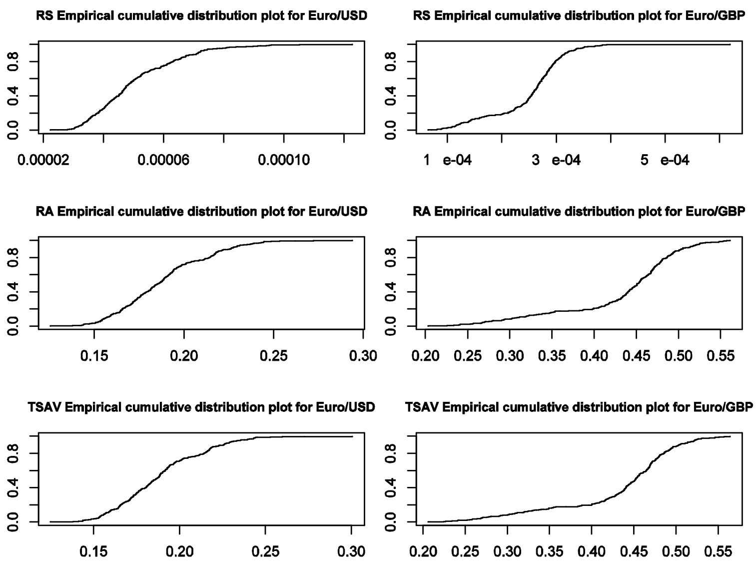
A simple comparison of realized volatilities, computed based on models (2.6, 1.5, and 2.10) for Euro/USD and Euro/GBP, with a traditional constant variance using Tables 4.2 and 4.3 detects that all realized measures tend to report volatility higher than a constant value. However in Table 4.3, the mean of RS realized volatility is smaller than that of two others.

Statistic	Euro/USD			Euro/GBP		
	$\langle \widehat{X,X} \rangle_{RS}$	$\langle \widehat{X,X} \rangle_{RA}$	$\langle \widehat{X,X} \rangle_{TSAV}$	$\langle \widehat{X,X} \rangle_{RS}$	$\langle \widehat{X,X} \rangle_{RA}$	$\langle \widehat{X,X} \rangle_{TSAV}$
Mean	5.06e-05	1.88e-01	1.87e-01	2.51e-04	0.434	0.434
Median	4.74e-05	1.86e-01	1.86e-01	2.65e-04	0.453	0.452
Variance	2.24e-10	6.80e-04	6.78e-04	4.79e-09	0.005	0.005
Skewness	1.09e+00	5.88e-01	6.11e-01	-2.95e-01	-1.12	-1.12
Kurtosis	1.68e+00	3.08e-01	4.81e-01	2.11e+00	0.697	0.691
Jarque-Bera	2.2e-16	2.6e-05	4.7e-06	9.9e-16	2.2e-16	2.2e-16

Table 4.3 Basic statistics and tests of realized volatility measures.

Based on exchange rate data, Andersen et al. (2001b) found that the distributions of realized daily variances are skewed to the right side and leptokurtic. In line with this finding, based on stock exchange data, Andersen et al. (2001a) also confirm that the unconditional distributions of realized variances are highly right-skewed. The volatilities of Euro/USD in Table 4.3 are rightward too. But all volatility measures in case of Euro/GBP show leftward skewness. A part of values of Euro/GBP rate, as can be observed in Figure 4.1, lies below the average for a while and will form leftward asymmetry. Four moments of realized volatility measures plus median are included in Table 4.3. Skewness and kurtosis of measures determine in more detail, none of the measures possess exactly a normal distribution. In terms of the Jarque-Bera test for normality reported in the table, none of measures hold normal distribution. With p-values equal to or smaller than 2.6e-05, normality for all measures is significantly rejected. However, a relative comparison may include informative facts. In case of Euro/USD rate, skewness coefficients for absolute based volatility measures are close together and closer to that of a normal distribution than that for squared based measure. Because of this, the normality in absolute based measures in Euro/USD rate is not rejected as strongly as in square based measure. The main reason for the difference among the distribution of volatility series may most likely be akin to different sensitivity to jumps. According to Andersen et al. (2001a) squared returns approach, over the relevant return horizon, provides model-free unbiased estimates of the ex post realized volatility. Unfortunately, however, squared returns are also a very noisy volatility indicator and hence do not allow for reliable inference regarding the true underlying latent volatility. Construction of realized volatility based on squared transformation seems not to be immune against jumps. In turn, this kind of transformation can be considered as a source of generating higher jumps in a series. In fact, squared based volatility measures reinforce jumps in original series. However, realized volatility constructed by absolute transformation seems relatively to be more monotonous. These arguments are also confirmed by Figure 4.2. The shapes show heavy tails. Presence of big jumps in squared based volatility is obviously evident in Figure 4.2. As such, these jumps lead the time series of measure to form a longer tail in distribution. The distribution holding the longer tail among others in Figure 4.2 is simply distinguishable. These jumps are the cause of greater positive skewness coefficient (to the right side) for Euro/USD in Table 4.3. Overall all daily time series of measures shape a kind of non-normal distribution, but absolute based series seem closer to normal. A part of these findings is in agreement with that of Andersen et al. (2001a). Of course, this phenomena was well documented as the fact of markets where the distribution of relative price changes is strongly non-Gaussian: these distributions can be characterized by power law tails with an exponent close to 3 for rather liquid markets. Emerging markets have even more extreme tails, with an exponent that can be less than two - in which case the volatility is infinite (Bouchaud, 2002). We will study this phenomenon in detail under dynamical properties of measures below.

Figure 4.2 Empirical cumulative distribution plots for Euro/USD and Euro/GBP seem skewed rightward and leftward respectively. However, the shapes are not the same. Asymmetry degree seems different among volatility series. Relative big positive jumps are present especially in RS volatility.



Since the most commonly used measure to analyze comovements and cointegration among international financial markets is correlation analysis; realized correlation is applied on 1 minute frequency exchange rates. Based on models (2.11, 2.12, and 2.15) our study is focused on correlation between the returns of the previous time series. In our analysis, both series belong to very developed, active and liquid markets. A main difference of our correlation with that of traditional analysis includes variation and hence likely dynamics of realized measure over time. In Figure 4.3, some distributional properties of different realized correlation measures are graphically embodied. First row plots explicitly imply that realized correlation series, against classical formulation of correlation, are time-varying, what is a profound property of many financial phenomena, and that they may have some dynamics. Their kernel density can be found in the second row of plots. As Andersen et al. (2001a) and Andersen et al. (2001b) reported the distributions of standardized realized squared correlation between 5 minute stocks and between 5 minute exchange rates are approximately normal. In our experiment here on 1 minute frequency data, the $RSCOR_{xy}$ and $RACOR_{xy}$ correlation series provide a normal distribution.

Figure 4.3 Distributional properties of realized correlations between Euro/USD and Euro/GBP are graphically embodied. Evidently realized correlations, based on first row plots, fluctuate over the time. The correlations oscillate almost around zero mean. The RS and RA based correlations possess a near symmetric density approximately with zero mean, while density of the other is positively skewed. These findings are more informatively supplemented by QQ-normal plots. Both RS and RA based correlations seem to shape a normal distribution.

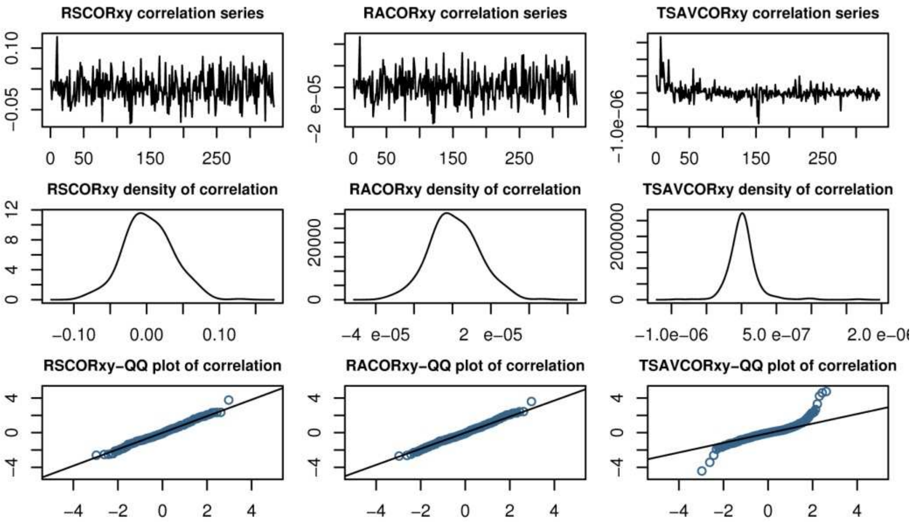


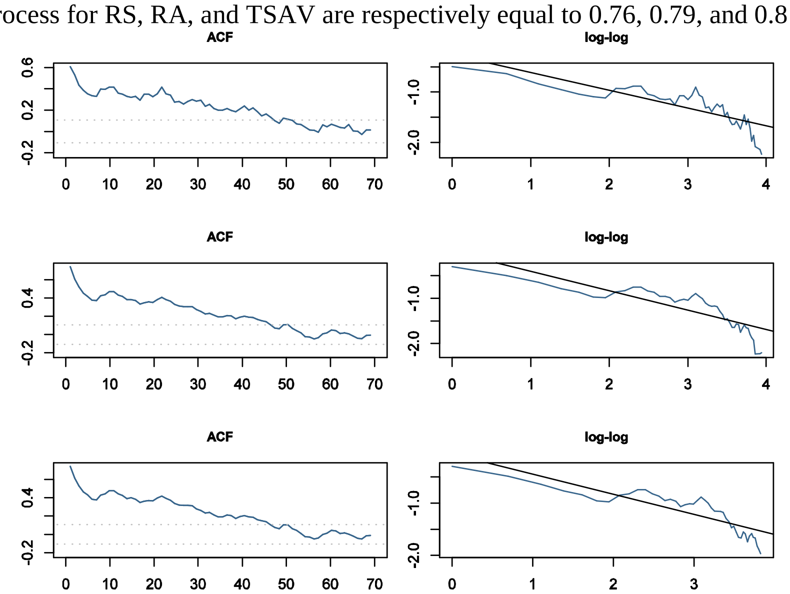
Table 4.4 reports some basic distribution-related statistics of realized correlations. All correlations have a positive mean. $RSCOR_{xy}$ in particular, shows the highest correlation between returns of the rates on average and hence the strongest degree of integration between markets over our time period. $RACOR_{xy}$ and $TSACOR_{xy}$ correlations behave relatively more stable over the time, since they have much less variance than $RSCOR_{xy}$ correlation. Comparing both mean and variance of different correlations, we observe that $RSCOR_{xy}$ correlation shows a stronger (based on mean value), and at the same time, more unstable (based on variance) relation between markets. Both $RSCOR_{xy}$ and $RACOR_{xy}$ correlations are slightly skewed to the right side. But rightward skewness of $TSACOR_{xy}$ measure is relatively considerable. Regarding to the Table 4.4, the p-values of Jarque-Bera test for null normality test are statistically significant at the 5 percent level for $RSCOR_{xy}$ and rather for $RACOR_{xy}$ correlations. Normality in $TSACOR_{xy}$ correlation series can not be significantly accepted.

Based on rather high skewness of $TSACOR_{xy}$ correlation, we found that positive asymmetry is present in the conditional realized correlation distribution. If relationship between markets complies the $TSACOR_{xy}$ correlation, then based on our data, upside comovements are greater than downside ones.

Statistic	$RSCOR_{xy}$	$RACOR_{xy}$	$TSACOR_{xy}$
Mean	2.75e-03	1.03e-06	1.84e-08
Median	-5.16e-04	-1.84e-07	8.44e-09
Variance	1.10e-03	1.61e-10	4.41e-14
Skewness	1.61e-01	1.60e-01	2.07e+00
Kurtosis	2.26e-01	1.68e-01	1.53e+01
Jarque-Bera test	0.315	0.379	2.2e-16

Table 4.4 Basic statistics and test of realized correlations.

Figure 4.4 Autocorrelation function and long memory autocorrelation function plots (ACF and log-log) of volatilities, computed based on returns on Euro/USD data. For all functions of both kind of autocorrelation function and long memory autocorrelation function, the number of lags is equal to 70. The top row belongs to the RS measure, the middle to RA, and the bottom to TSAV. Left plots are autocorrelation functions and right ones are long memory autocorrelation functions. The estimated Hurst exponents (self-similarity parameter) in the long memory process for RS, RA, and TSAV are respectively equal to 0.76, 0.79, and 0.81.

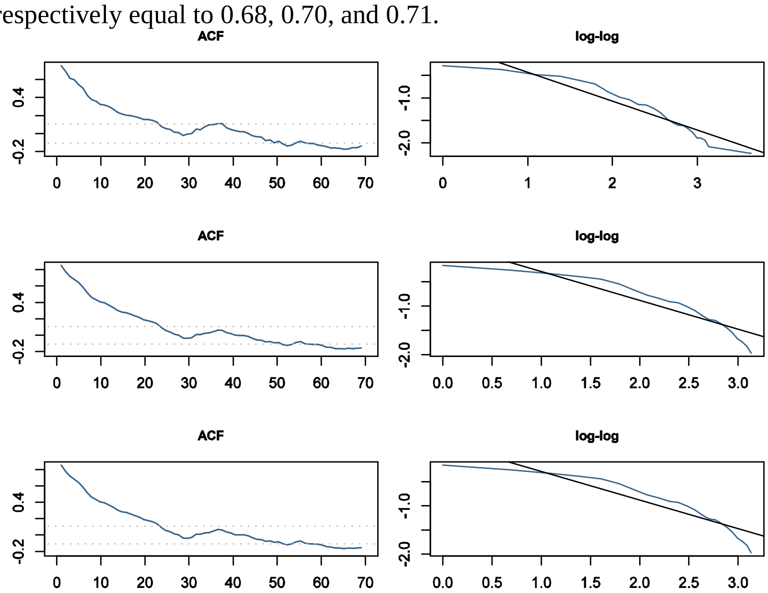


4.1. Dynamic Behaviour of Volatilities and Correlations

Behavior analysis of the estimators, for example, study of stylized facts of financial time series could be interesting and informative. For some useful information about several stylized facts refer to Cont (2001). Now, issues related to dynamic behaviors of measures are extracted by detailed examinations with particular focus on the long memory and scaling law.

In Figures 4.4 (for Euro/USD) and 4.5 (for Euro/GBP), on the left panel, autocorrelation function (ACF) plots and on the right panel, long memory plots for realized volatilities have been drawn. Ding et al. (1993) and Andersen and Bollerslev (1997) have argued that the autocorrelations of squared and absolute returns decay at a much slower hyperbolic rate over longer lags. Consistent with these authors, the figures almost identically indicate a slow decay in autocorrelation over time for all measures. Long memory may be a very interesting signature for series dynamics. Usually it is spoken of a long memory behavior, if the decay in the ACF is slower than a hyperbolic rate, i.e. the correlation function decreases algebraically with increasing (integer) lag. Thus it makes sense to investigate the decay on a double logarithmic scale and to estimate the decay exponent. Graphically, if the time series exhibits long memory behavior, it can easily be observed as a straight line in plot on the right panels of Figures 4.4 and 4.5. Corresponding long memory plots of volatility series in Figures 4.4 and 4.5 show a slow decay for measures, meanwhile absolute based measures indicate longer memory numerically estimated by Hurst exponents which will explained below. So, the volatility measures include long memory behavior as a dynamic stylized fact of market. This finding at 1 minute frequency is consistent with those empirical experiments on tickers included in NASDAQ by Andersen et al. (2001a) and in DM/US dollar and Yen/US dollar exchange rates by Andersen et al. (2001b) both in 5 minute frequency.

Figure 4.5 Autocorrelation function and long memory autocorrelation function plots of volatilities, computed based on returns on Euro/GBP data. For all functions of both kind of autocorrelation function and long memory autocorrelation function, the number of lags is arbitrarily equal to 70. The top row belongs to RS measure, the middle to RA, and the bottom to TSAV. Left plots are autocorrelation functions and right ones are long memory autocorrelation functions. Estimated Hurst exponent (self-similarity parameter) in long memory process for RS, RA, and TSAV are respectively equal to 0.68, 0.70, and 0.71.



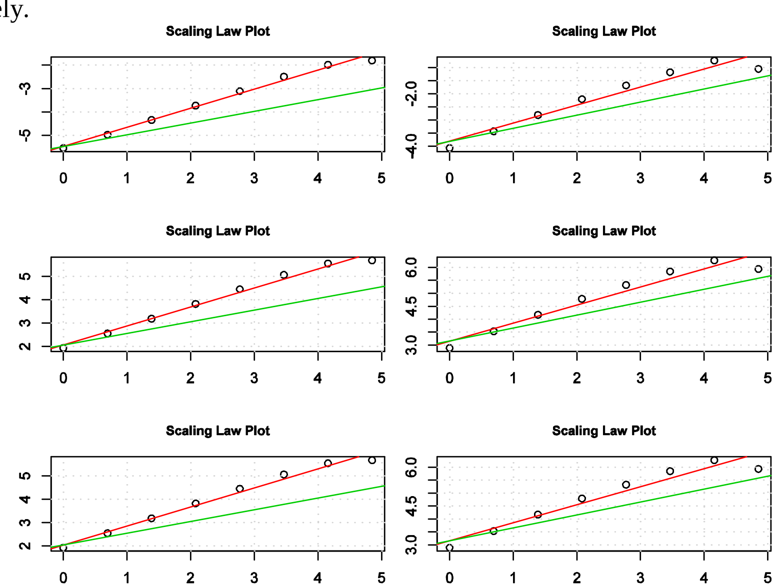
Another striking fact of markets is the regular fractal structure of the financial series in the sense of Mandelbrot (1986). This is illustrated by the scaling laws usually reported for the volatility time series under aggregation. The scaling law for the volatility relates the volatility over a time interval to the size of this interval. In other words, considering the average absolute return over individual data periods, one finds a scaling power law which relates the mean volatility over given time intervals to the size of these intervals. The power law is in many cases valid over several orders of magnitude in time. Its exponent usually deviates significantly from a Gaussian random walk model which implies 0.5. This other implication of self-similarity and long memory associated with fractional integration concerns the behavior of variance of partial sums. In particular, let $[x_t]_t \equiv \sum_{j=1,\ldots,h}\} x_{h.(t-1)+j}$, denote the h-fold partial sum process for x_t, where t=1,2,...,[T/h]. Then, if x_t is fractionally integrated, the partial sums obey a scaling law,

$$Var([x_t]_h) = c \cdot h^{2d+1} \tag{4.20}$$

where c is a constant, and d is scaling parameter. The variance of realized volatility should grow at rate h^{2d+1}. Scaling parameter refers to the elasticity of volatility series with respect to the timescale. Estimated parameters for Euro/USD are equal to 2.04, 2.14, and 2.13; and for Euro/GBP to 1.94, 1.96, and 1.97 in the structures of RS, RA, and TSAV volatility measures respectively. Figure 4.6 illustrates that all volatilities in Euro/USD and Euro/GBP follow a regularity based on which log variance of partial sum proportional to log variance of the whole period; and that the plots of scaling law for volatilities are almost similar to each other.

Like for the Hurst exponent, this regularity also stimulates one to think of predictability in financial markets.

Figure 4.6 A scaling law plot of realized volatilities displays regularity based on which partial sums of volatility against the time intervals follow the scaling law. This regular behavior is also considered as a statistical self-similarity in volatilities time series. The left panel indicates Euro/USD and the right one, Euro/GBP. First, middle, and bottom rows belong to RS, RA, and TSAV volatilities respectively. Since all points in plots are close to the red line, scaling law exists in all volatilities. Estimated parameters for Euro/USD are equal to 2.04, 2.14, and 2.13; and for Euro/GBP equal to 1.94, 1.96, and 1.97 in the structures of RS, RA, and TSAV volatility measures respectively.



A self-similar series statistically means that the statistical properties for the entire data set are the same as for sub-sections of the data set. In other words, the self similar dimension of fractional integration is invariant to the horizon. From the slope of log-log plot in Figures 4.5 and 4.6, an exponent called Hurst exponent is derived. Usually the Hurst exponent is considered as the statistical self-similarity parameter (dimension) in the structure of a financial time series. The Hurst exponent, H, can be defined as H:=log(R/S)/log(T), where T is the duration of the sample of data, and R/S is the corresponding value of rescaled range. In this way, Hurst (1951) and Hurst (1955) generalized an equation valid for the Brownian motion process in order to include a broader class of time series. In fact, Einstein studied the properties of the Brownian motion and found that the distance R covered by a particle undergoing random collisions is directly proportional to the square-root of time T:

$$R = k{\cdot}T^{0.5}$$

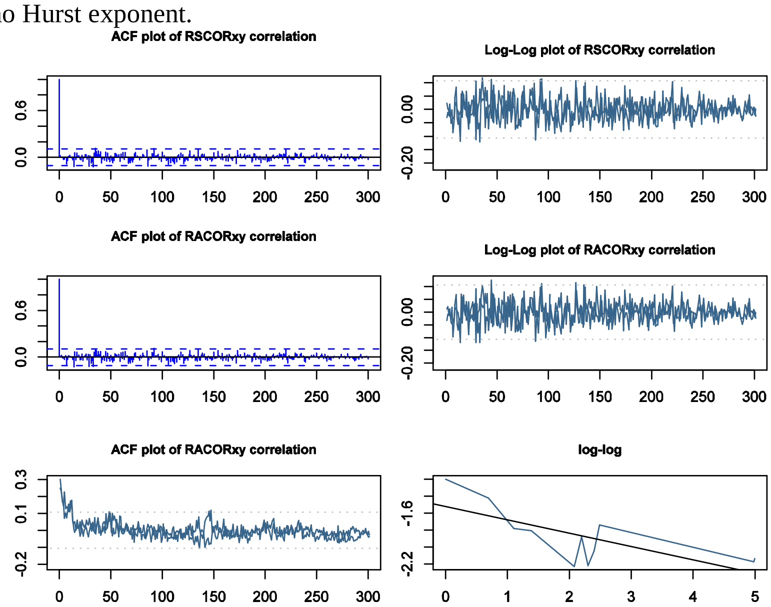
where k is a constant which depends on the time series. The generalization proposed by Hurst was

$$R/S= k{\cdot}T^{H} \tag{4.21}$$

where H is the Hurst exponent. Estimating the Hurst exponent for a data set provides a measure of whether the data is a pure random walk or has underlying trends. The values of the Hurst exponent range between 0 and 1. A Hurst exponent value within a range of $0.5 < H < 1$ indicates persistent behavior (e.g., a positive autocorrelation and hence a long memory). Furthermore, the closer H is to 1, the stronger the dependence of the process is. Data sets like this are sometimes referred to as fractional Brownian motion. A value of 0.5 indicates a true random walk (a Brownian time series with no autocorrelation). The fractal dimension is directly related to the Hurst exponent for a statistically self-similar data set. In a random walk there is no correlation between any element and a future element. A small Hurst exponent has a higher fractal dimension and a rougher surface. A larger Hurst exponent has a smaller fractional dimension and a smoother surface. A Hurst exponent value $0 < H < 0.5$ will exist for a time series with anti-persistent behavior (or negative autocorrelation). Here an increase will tend to be followed by a decrease and inversely. This behavior is sometimes called mean reversion. There are many estimators that are used to estimate the value of the Hurst parameter[4]. Estimated Hurst exponents by R/S method are equal to 0.76, 0.79, and 0.81 for Euro/USD and to 0.68, 0.70, and 0.71 for Euro/GBP in the structure of RS, RA, and TSAV volatility measures respectively. As an example in a simulation study for an artificial capital market, the Hurst exponent for the prices generated by the trading of the agents is estimated between 0.65 and 0.71 (Schlottmann and Seese, 1999). In fact, there is the strong evidence to suggest that volatility is a long memory process, consistent with Andersen et al. (1999a).

An investigation of the fact that if the patterns and temporal dependencies of comovements across equity markets behave regularly, can help here too. Existence of such regularities imply the dynamics of correlation series. We are now interested to find regular patterns in correlations, if there are any. Considering Figure 4.7, a long autocorrelation (ACF plot) in the structure of $RSCOR_{xy}$ and $RACOR_{xy}$ has been now completely disappeared. Based on the long memory autocorrelation plot in Figure 4.7, a temporal dependence for $RSCOR_{xy}$ and $RACOR_{xy}$ can not be reported. Of course, $TSACOR_{xy}$ seems to keep still its dynamic properties. It exhibits the long memory dependence with Hurst exponent equal to 0.92.

Figure 4.7 Autocorrelation function and long memory autocorrelation function plots (ACF and log-log) of correlations between Euro/USD and Euro/GBP. For all functions of both kind of autocorrelation function and long memory autocorrelation function, the number of lags is equal to 300. The top row belongs to $RSCOR_{xy}$, the middle to $RACOR_{xy}$, and the bottom to $TSACOR_{xy}$ correlation. Left plots are autocorrelation functions and right ones are long memory autocorrelation functions. The estimated Hurst exponent (self-similarity parameter) in long memory plot for $TSACOR_{xy}$ is equal to 0.92. $RACOR_{xy}$ and $RSCOR_{xy}$ exhibit no long memory and consequently have no Hurst exponent.

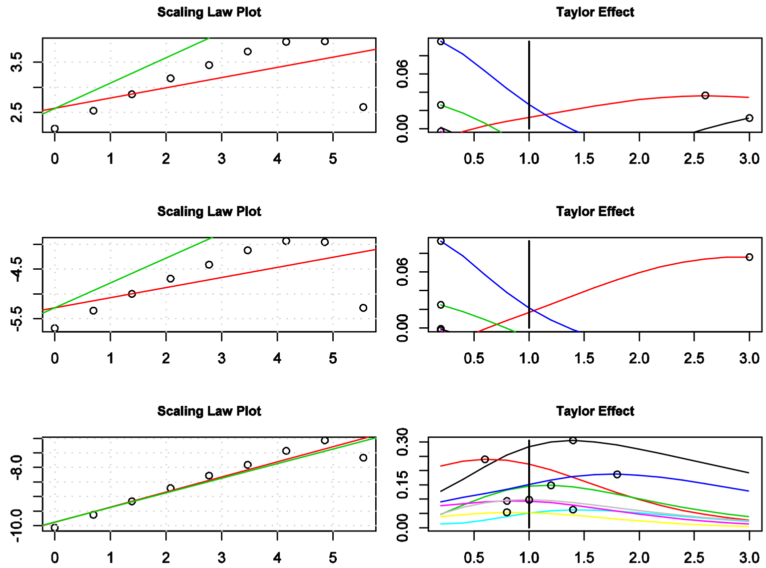


The calculated points in scaling law plots for $RSCOR_{xy}$ and $RACOR_{xy}$ correlations are far from the estimated red line in Figure 4.8. It is not possible to fit a straight line which links all points and hence the corresponding plots can not show the scaling law. However, the plot related to $TSACOR_{xy}$ correlation shows well scaling law property with scaling parameter equal to 1.93.

[4] Some more common methods include Absolute value method, Variance method, R/S method, Periodogram method, Whittle estimator, Variance of residuals, and Abry-Veitch method.

Analyzing 40 series of returns, Taylor (1986) observes that the sample autocorrelations of absolute returns seem to be larger than the sample autocorrelations of squares. Let Y_t, t=1,...,T be the series of returns and $r_\theta(k)$ denotes the sample autocorrelation of order k of $|y_t|^\theta$, $\theta > 0$; the Taylor Effect can be defined as $r_1(k) > r_\theta(k)$ for any $\theta \neq 1$. The autocorrelations of absolute returns to the power of theta reach their maximum at $\theta = 1$. In Figure 4.8, plots display autocorrelations as a function of the exponent θ for each lag from 1 to the maximum lag (e.g., 10 lags). In case that the above formulated hypothesis is supported, all the curves should peak at the same value around $\theta = 1$. Figure 4.8 indicates that none of the curves in corresponding plots for $RSCOR_{xy}$ and $RACOR_{xy}$ correlations reach their pinnacle around $\theta = 1$ and the points are distant from vertical line of $\theta = 1$. In contrast, the plot related to the $TSACOR_{xy}$ measure exhibits somewhat Taylor Effect.

Figure 4.8 According to scaling law plots in left panel, $TSACOR_{xy}$ has a high performance of dynamics. The points on a scaling plot for $RSCOR_{xy}$ and $RACOR_{xy}$ correlations are far from the estimated line and hence they can not show the scaling law. The estimated exponent is equal to 1.93 in $TSAVCOR_{xy}$ correlation series. The Taylor effect plot indicates that Taylor Effect exists in a series, where the curves peak at the value around θ=1 which is on the x axis. Top, middle, and bottom rows belong to $RSCOR_{xy}$, $RACOR_{xy}$ and $TSAVCOR_{xy}$ correlations respectively. This effect is present in $TSACOR_{xy}$ correlation regarding to the number of lags which is arbitrarily selected to be equal to 8. In the $TSACOR_{xy}$ correlation, the Taylor Effect plot peaks around θ=1 with 1 lag against with no lag for both other correlations



The Hurst exponent and scaling law promise a gleam of hope for predictability in financial markets which seemingly sound unpredictable at all, under the efficient market hypothesis; since they show well regularity in chaotic and stochastic behaviors of particles or agents. Peters (1996) suggests that a Hurst exponent value between $0.5 < H < 1.0$ shows that the efficient market hypothesis is incorrect. Returns are not randomly distributed. There is some underlying predictability. But the problem of estimating the Hurst exponent itself, involves a complex problem of accurate calculation. Moreover, we are not certain about a especial variable of interest to be a representative for predictability of the market. In our investigation here, volatility reflects regularity in market. But as reported by many, for example Ding et al. (1993), original prices do not show such the regularity, at least by Hurst exponent, among statistics. It is now well established that the stock market returns themselves contain little serial correlation which is in agreement with the efficient market theory. But this empirical fact does not necessarily imply that returns are independently identically distributed as many theoretical financial models assume. It is possible that the series is serially uncorrelated but is dependent. For the stock market data is especially so, since if the market is efficient, a stock's price should change with the arrival of information. If information comes in bunches, the distribution of the next return will depend on previous returns although they may not be correlated. As the return period increases, the return values reflect longer trends in the time series. Perhaps the higher Hurst exponent value is actually showing the increasing upward or downward trends. This does not, by itself, show that the efficient market hypothesis is incorrect. Even if we accept the idea that a non-random Hurst exponent value does damage to the efficient market hypothesis, estimation of the Hurst exponent seems of little use when it comes to time series forecasting. At best, the Hurst exponent tells us that there is a long memory process. The Hurst exponent does not provide the local information needed for forecasting. Nor can the Hurst exponent provide much of a tool for estimating periods that are less random, since a relatively large number of data points are needed to estimate the Hurst exponent. For example a constant Hurst exponent over time also does not seem a sound and reasonable conclusion. However, this statistic can be useful in analyzing the behavior of market models.

5. RELATIONSHIP BETWEEN VOLATILITY AND CORRELATION

A study on multivariate relationship between estimators, in particular, volatility and correlation estimators, can help to figure out whether and how *TSAV*, and $TSACOR_{xy}$ move together. Such questions are difficult to answer using conventional volatility models, but they are relatively easy to address using the time-varying realized volatilities and correlations. A strong evidence has been observed that realized volatilities and correlations move together. Realized correlation is itself correlated with realized volatility, which is called the volatility effect in correlation (*VIC*) (Andersen et al., 2001a).

Andersen et al. (2001b) estimate a kernel density of relationship between realized correlation and logarithmic realized standard deviation when the medians of both logarithmic realized standard deviations of Deutsche Mark and Yen are less than a threshold equal to -0.46 and when both are greater than -0.46 and show density distributions of high volatility days differ from that of low volatility days. Huang and Nieh (2004) approximate a linear regression and show a positive relationship between realized correlation and volatilities significantly. To do this task, we have to turn back to the conventional techniques which fail to formulate directly observable instantaneous and contemporaneous relationship. We intend to estimate a simple linear least square regression. It is assumed that the realized correlation follows the realized volatility. In Table 4.5, the results of linear regression estimation are reported. In an experiment, the realized correlation between realized volatility of returns on Euro/USD and on Euro/GBP is modeled to follow the realized volatility of returns on Euro/USD and in another experiment, on Euro/GBP exchange rate. Different estimators of volatility and correlation are considered.

In the first experiment, different results in terms of the type and intensity of the relationship were obtained, while P-values for parameters *a* (constant value) and *b* (slope) for three estimators, particularly in case of *TSAV*, are high. The relationship between realized correlation and volatility in case of Euro/USD rate, estimated to be negatively strong (-67.93 for parameter *b*) based on *RS* estimator and to be negatively mild (-2.2e-05) based on *RA* estimator. Meanwhile, the relationship is reported to be positive (6.1e-07) by *TSAV* estimator. As a matter of fact, according to the latter relationship, when the Euro/USD exchange market is highly volatile (measured by realized volatility), the relationship (measured by realized correlation) between the two markets (Euro/USD and Euro/GBP) becomes stronger, and when the Euro/USD market goes to calm down, the association between the markets goes to relax. So, two markets tend to be highly correlated when the Euro/USD market is highly volatile and inversely. A similar correlation effect in volatility was documented for international equity returns by Solnik et al. (1996).

Model	RS		RA		TSAV	
	Parameter	P-value	Parameter	P-value	Parameter	P-value
Corr=f(vol. of returns on Euro/USD):parameter a	0.0062	0.33	5.1e-06	0.31	-3.4e-08	0.78
Corr=f(vol. of returns on Euro/USD):parameter b	-67.93	0.56	-2.2e-05	0.42	6.1e-07	0.56
Corr=f(vol. of returns on Euro/GBP):parameter a	0.0058	0.39	3.1e-06	0.46	4.6e-07	7.4e-07
Corr=f(vol. of returns on Euro/GBP):parameter b	-12.21	0.64	-4.8e-06	0.62	-1.7e-06	3.2e-06

Table 4.5 Results of regression estimation: Correlation as a function of volatility.

In the second experiment, it is assumed that the correlation between Euro/USD and Euro/GBP is associated with the volatility in Euro/GBP exchange market. As the table reports, existence of such the relationship is rejected by the two-scale measure, since the p-values are very small (for example 3.2e-06 for parameter *b*). However, a negative relationship in terms of *RS* and *RA* measures is meaningfully approximated.

6. CONCLUSION AND DISCUSSION

The study of some important distributional and dynamic aspects of different alternative realized volatility and correlation measures was the score of the present article. The distribution of realized squared volatility tends to be highly rightward skewed. The two-scale realized absolute volatility measure is so formulated that more accuracy and less bias is additionally added to the realized absolute volatility measure by inclusion of sampling and averaging procedure while applying higher frequency data contaminated by microstructure noise. Here, market microstructure noise is effectively damped by constructing K series of aggregate returns of K samples which are then used to compute K intermediate and inconsistent estimators that will be averaged to obtain, at last, the desired consistent estimator and to be improved by bias-corrector term. Likewise, the Jackknife method resamples the statistic at the *n* points. The estimators investigated in this paper are constructed based on the subsampling method. Goncalves and Meddahi (2005) propose bootstrap methods for statistics evaluated on high frequency data such as realized volatility. However, application of other bias-corrector methods, in particular, the Jackknife method is worthy to investigate further in the area of realized volatility. A comparison of different methods for bias-correction may reveal some valuable results.

Regarding to our 1 minute data of exchange rates, a comparison of different volatility measures suggests that daily realized absolute based volatilities appear closer to the normal distribution relative to realized squared based volatility. However, none of investigated measures absolutely pose a normal daily distribution tested by Jarque-Bera test of normality. In our experiments, we found that absolute based volatility measures include longer memory behavior as a dynamic stylized fact of markets, although squared based measure exhibits long memory behavior too. Self-similarity structures computed by the Hurst exponent and regular fractal scaling law were documented in the structures of series generated by realized measures.

The normality of two-scale based correlation can not be accepted. But the realized squared and absolute correlations are viewed to pose a shape of the normal distribution, and in terms of the Jarque-Bera test the normality can not be rejected. According to our experiment, two latter correlations seem to fail containing dynamic properties such as long memory as well as scaling law. While two-scale based correlation measure suffers from non-normality, autocorrelation, long memory and scaling law, which have been well documented in real world time series processes, are included in its structure. This may mean predictability in the market by this measure. According to our empirical work, we could document statistical self-similarity dimension estimated by a Hurst parameter as well as a fractal structure illustrated by scaling law as another implication of self-similarity structure in our $TSACOR_{xy}$ correlation measure. Strong positive asymmetry in $TSAVCOR_{xy}$ correlation implies that upside co-movements are greater than downside comovements between markets.

Time-varying volatility and correlation measures offer a good tool for more profound analysis of, for example, association between volatilities and correlations. We found that when the Euro/USD market is highly volatile, relationship between the Euro/USD and Euro/GBP becomes stronger, and when the Euro/USD time series goes to calm down, the association between the markets goes to relax.

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